MATH 211 PRACTICE EXAM

January 24, 2024 (3 hours)

No books, notes, or electronic devices (especially phones) are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Do not unstaple the exam or reorder the pages! All problems must be solved within the space provided (right after the statement of the problem). If you need to use the extra pages at the end, then mention this clearly in the aforementioned space, so your grader knows that they have to also look at the end (they will not do so otherwise).

Please do not leave the room during the first half hour and the last hour of the exam.

There are 8 problems, worth 100 points in total.

NAME: ______________________________

SCIPER: ______________________________

PROBLEM 1

(a) If G is a group such that the function

$$f : G \to G, \qquad f(g) = g^2, \ \forall g \in G$$

is a homomorphism, then prove that G is abelian. *(4 points)*

(b) Prove that the dihedral group D_{24} is not isomorphic to the symmetric group S_4. *(4 points)*

(c) Prove (by checking all the axioms) that the set of invertible 2×2 real matrices forms a group with respect to matrix multiplication. *(4 points)*

PROBLEM 2

(a) If H is a subset of a group G, then the normalizer of H is defined as

$$N_G(H) := \{g \in G | gHg^{-1} = H\}$$

Show that if G is finite, then the definition above is equivalent to the weaker property

$$N_G(H) = \{g \in G | gHg^{-1} \subseteq H\}$$

(5 points)

(b) Find, with proof, the center of the group S_5. *(5 points)*

(c) Suppose that a group G of order n has class equation (i.e. sum of the sizes of its conjugacy classes) given by

$$n = \underbrace{1 + \cdots + 1}_{d \text{ times}} + a_1 + \cdots + a_k$$

with $a_1, \ldots, a_k > 1$. Prove that the number d of 1's divides $\frac{n}{a_1}, \ldots, \frac{n}{a_k}$.

(partial credit if you tell us the meaning of d in terms of the group G). *(8 points)*

PROBLEM 3

Suppose we have a <u>transitive</u> action of a finite group G on a finite set X.

(a) The orbit-stabilizer theorem applied to this action is an equality of natural numbers that involves the stabilizer H of any given $x \in X$. Write this equality down in the specific case of the given <u>transitive</u> action $G \curvearrowright X$. *(6 points)*

(b) If the above transitive action $G \curvearrowright X$ is also free, then

$$|X| = \dots$$

and there exists a bijection

$$X \cong \underline{\hspace{2em}}$$

with respect to which the given action $G \curvearrowright X$ corresponds to the

$$- - - - \text{ action } G \curvearrowright \underline{\hspace{2em}}$$

Fill in the dots, blanks and dashes with stuff that depends only on G. *(6 points)*

PROBLEM 4

(a) Prove or disprove (via a counterexample) the following statement: if finite groups K, L and G have the property that there exists a short exact sequence

$$1 \to K \to G \to L \to 1$$

then there also exists a short exact sequence

$$1 \to L \to G \to K \to 1$$

(6 points)

(b) Same question as above, but assuming $G = \mathbb{Z}/n\mathbb{Z}$ for some natural n. *(6 points)*

PROBLEM 5

(a) Suppose A, B, C are simple groups, none of which is isomorphic to a subgroup of any one of the others. How many composition series does the group $A \times B \times C$ have? Fully justify your answer. *(6 points)*

(b) Calculate the n-th torsion subgroup of $\mathbb{Q}/\mathbb{Z}$ (the rational numbers modulo the integers, made into a group by addition)? *(8 points)*

PROBLEM 6

Show that any finite p-group is G super-solvable, i.e. it has a series of subgroups

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_{k-1} \lhd G_k = G$$

with each G_i normal in G and each quotient G_{i+1}/G_i being cyclic. *Hint: you may use without proof the fact that the center $Z(G)$ of a p-group G is non-empty.* *(10 points)*

PROBLEM 7

(a) Show that if H is a normal p-subgroup of a finite group G, then H is contained in any Sylow p-subgrup of G. *(harder, 6 points)*

(b) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Prove that G has a normal Sylow 11-subgroup P and that $P \subseteq Z(G)$. *Hint: consider the conjugation action of* G *on* P. *(harder, 8 points)*

PROBLEM 8

We saw in class that a finite abelian group has as many one-dimensional representations (with complex coefficients) as its order. How many one-dimensional representations does a general finite group G has? Answer, with proof, in terms of the commutator subgroup $[G, G]$. *(8 points)*

Partial credit if you can describe the one-dimensional representations of $G = \mathbb{Z}/n\mathbb{Z}$.

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